ε-MAPPINGS ONTO POLYHEDRA

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1. **Introduction.** All spaces in this paper are assumed to be metric. A compactum is a compact metric space and a continuum is a connected compactum. By a polyhedron we mean a triangulable compactum.

Given a positive real number $\varepsilon > 0$, a compactum X with metric d and a (continuous) mapping $f: X \to Y$ onto Y, f is said to be an *ε-mapping* provided, for each $y \in Y$, the diameter diam $f^{-1}(y) \leqq \varepsilon$. This notion is well known and has been introduced by P. S. Aleksandrov in 1928 [1].

DEFINITION 1. Let $\Pi = \{P_\alpha\}$, $\alpha \in A$, be a class of polyhedra. We say that a compactum X is Π-like (or like Π) provided, for each $\varepsilon > 0$, there is a polyhedron $P_\alpha \in \Pi$ and an ε-mapping $f: X \to P_\alpha$ onto P_α.

Clearly, the notion is independent of the choice of the metric d on X.

We shall denote the set of all Π-like compacta by (Π). Clearly, $\Pi \subset (\Pi)$.

EXAMPLE 1. If Π consists of all polyhedra (connected polyhedra), then (Π) consists of all compacta (continua).

This is readily proved by using geometric realizations of the nerves of open coverings and by applying canonical mappings into nerves (e.g. cf. [7, Theorem 11.8, p. 286]).

EXAMPLE 2. If Π consists of all polyhedra P with $\dim P \leqq n$, then (Π) consists of all compacta X with $\dim X \leqq n$. This is Aleksandrov's theorem characterizing dimension by approximation by polyhedra [2].

EXAMPLE 3. If Π consists of all acyclic graphs, then (Π) is the class of tree-like continua [4].

EXAMPLE 4. An important case is obtained when $\Pi = \{P\}$ consists of a single polyhedron P. For instance, if P is the arc $I = [0, 1]$, then (Π) is the class of snake-like continua [4]. Especially interesting is the case of M-like continua, where M is a (triangulable) connected manifold. This class has recently been studied by several authors, in particular by M. K. Fort [8], T. Ganea [11; 12] and A. Deleanu [5].

REMARK 1. Definition 1 can be generalized by allowing Π to be any class of compacta, not necessarily triangulable. In particular, if Π consists of a single

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compactum Y, we encounter the order relation introduced by C. N. Maxwell in [16].

In the present paper we first prove the existence of an expansion of Π-like continua into inverse sequences of polyhedra $P_\alpha \in \Pi$ (see Theorem 1). Next, applying these expansions we obtain representations of locally cyclic Π-like continua X as limits of sequences of polyhedra $P_\alpha \in \Pi$, which converge completely avoidably 0-regularly to X in the sense of P. A. White [18] (see Theorem 3). This enables us to apply known results on regular convergence, thus obtaining certain new results on M-like continua, where M is a closed orientable 2-manifold (see Theorem 4).

2. **Expanding Π-like continua into inverse sequences of polyhedra from Π.** An inverse sequence $\{X_i; \pi_{ij}\}$ is a sequence of compacta X_i, $i = 1, 2, \cdots$, and mappings $\pi_{ij}: X_j \to X_i$ onto X_i, $i \leqq j$, such that π_{ii} is the identity and

$$\pi_{ij}\pi_{jk} = \pi_{ik}, \qquad i \leqq j \leqq k. \tag{1}$$

The maps π_{ij} are called bonding maps and it is essential in this paper that they be mappings onto. The corresponding inverse limit $X = \mathrm{Inv}\lim\{X_i; \pi_{ij}\}$ is the subset of the product space $Q = X_1 \times X_2 \times \cdots$ consisting of all points $x \in Q$, for which

$$\pi_{ij}\pi_j(x) = \pi_i(x), \qquad x \in X,\ i \leqq j; \tag{2}$$

here $\pi_j: Q \to X_j$ denote the natural projections. Clearly, $\pi_j: X \to X_j$ are mappings onto. It is well known that X is a compactum(2).

The first relation between the notion of being Π-like and inverse limits is established by the following simple lemma.

LEMMA 1. *Let Π be a class of polyhedra and $\{P_i;\ \pi_{ij}\}$ an inverse sequence of polyhedra $P_i \in \Pi$ with maps π_{ij} onto. Then the inverse limit X is a Π-like compactum.*

Proof. Given $\varepsilon > 0$, choose an open covering $\mathscr{U} = \{U^1, \cdots, U^n\}$ of X with $\operatorname{diam} U^i \leqq \varepsilon$. We can assume that each U^i is of the form

$$U^i = \pi_{k(i)}^{-1}\ (V_{k(i)}), \tag{3}$$

where $V_{k(i)}$ is open in $X_{k(i)}$. Let $k = \mathrm{Max}\{k(1), \cdots, k(n)\}$ and let

$$V_k^i = \pi_{k(i)k}^{-1}(V_{k(i)}). \tag{4}$$

Then clearly

$$\pi_k^{-1}(V_k^i) = \pi_k^{-1}\pi_{k(i)k}^{-1}(V_{k(i)}) = \pi_{k(i)}^{-1}(V_{k(i)}) = U^i, \tag{5}$$

(2) For basic properties of inverse limits cf., e.g., Chapter VIII of [7].

because of

$$\pi_{k(i)} = \pi_{k(i)k}\pi_k. \tag{6}$$

The open sets V_k^i, $i=1,\cdots,n$, form an open covering of X_k. Indeed, if $x_k \in X_k$, then there is a point $x \in X$ with $\pi_k(x) = x_k$ (recall that π_k is onto). Choose $i \in \{1,\cdots,n\}$ such that $x \in U^i$. Then (3) implies

$$\pi_{k(i)}(x) \in V_{k(i)}, \tag{7}$$

and since $\pi_{k(i)}(x) = \pi_{k(i)k}\pi_k(x) = \pi_{k(i)k}(x_k)$, we infer that

$$x_k \in \pi_{k(i)k}^{-1}(V_{k(i)}) = V_k. \tag{8}$$

Now consider the mapping $\pi_k : X \to X_k$ which maps X onto X_k. π_k is an ε-mapping. Indeed, for $x_k \in X_k$, there is a member V_k^i of the covering $\{V_k^1,\cdots,V_k\}$ of X_k which contains x_k, and then by (5),

$$\pi_k^{-1}(x_k) \subset \pi_k^{-1}(V_k^i) = U^i, \tag{9}$$

so that

$$\operatorname{diam} \pi_k^{-1}(x_k) \leqq \operatorname{diam} U^i \leqq \varepsilon. \tag{10}$$

This completes the proof of Lemma 1.

Now we ask for the converse of Lemma 1 and shall prove

THEOREM 1. *Every* Π-like *continuum X is the inverse limit of an inverse sequence $\{P_i;\ \pi_{ij}\}$ with bonding maps π_{ij} onto and with polyhedra $P_i \in \Pi$.*

Combining Theorem 1 and Lemma 1 we obtain

THEOREM 1*. *Let Π be a class of connected polyhedra. Then the class of Π-like compacta coincides with the class of limits of inverse sequences $\{P_i;\pi_{ij}\}$ with maps π_{ij} onto and with $P_i \in \Pi$.*

REMARK 2. If Π consists of connected polyhedra, then all $X \in (\Pi)$ are continua. In Theorem 1 it is not possible to omit the assumption that X be connected as is shown by the following example.

EXAMPLE 5. Let P_n denote the polyhedron which consists of an arc $I_n = [0,1]$ and n isolated points, and let $\Pi = \{P_0, P_1, P_2, \cdots\}$. Let C denote the Cantor triadic set. It is readily seen that C is Π-like. However, if $\{X_i;\pi_{ij}\}$ is an inverse sequence with maps π_{ij} onto and with $X_i = P_{n(i)}$, we have $\pi_{ij}(I_{n(j)}) = I_{n\,i)}$ and thus the inverse limit X contains a subset $Y = \operatorname{Inv}\lim \{I_{n(i)};\pi_{ij}\}$, which is a nondegenerate snake-like continuum and has dimension 1. Therefore, X cannot be the Cantor set C.

If we let Π be the class of n-dimensional polyhedra, then Theorem 1 yields this

COROLLARY 1. *Every n-dimensional continuum X is the inverse limit of some inverse sequence of n-dimensional polyhedra with bonding maps onto.*

A stronger form of this result is a well-known theorem of H. Freudenthal [10].

3. **Proof of Theorem 1.** The proof is based on several lemmas.

LEMMA 2. *Let X be a compactum with metric d, P_1 a polyhedron with a fixed triangulation K_1 and metric d_1, and let $f_1 : X \to P_1$ be an ε_1-mapping onto P_1. Then there is an $\varepsilon_2 > 0$ such that, for any polyhedron P_2 and ε_2-mapping $f_2 : X \to P_2$ onto P_2, there exists a (simplicial) mapping $\phi : P_2 \to P_1$ (into P_1) having the property that $\phi(P_2)$ is the carrier of a subcomplex of K_1 and the open star $\mathrm{St}_{K_1}(\phi(P_2)) = P_1$. Moreover, if α_1 denotes the mesh of K_1, then $d_1(f_1, \phi f_2) \leqq 2\alpha_1$.*

Proof. Let $\beta > 0$ be a number (the Lebesgue number) such that every set $Q \subset P_1$ with diameter $\operatorname{diam} Q \leqq \beta$ is contained in the open star $\mathrm{St}_{K_1}(a)$ of some vertex a of K_1. Choose $\gamma > 0$ in such a manner that for any set $R \subset X$ with $\operatorname{diam} R \leqq \gamma$ one obtains $\operatorname{diam} f_1(R) \leqq \beta$ (f_1 is uniformly continuous). Then put $\varepsilon_2 = \gamma/2$.

Now assume that P_2 is a polyhedron and $f_2 : X \to P_2$ an ε_2-mapping onto P_2. It is readily seen (proof by contradiction) that there exists a $\delta > 0$, such that, for any set $S \subset P_2$ the relation $\operatorname{diam} S \leqq \delta$ implies $\operatorname{diam} f_2^{-1}(S) \leqq 2\varepsilon_2 = \gamma$. Now take a triangulation K_2 of P_2 so fine that its mesh α_2 satisfies $\alpha_2 \leqq \delta/2$. Clearly, for any vertex b of K_2, the diameter of $\mathrm{St}_{K_2}(b)$ is not greater than $2\alpha_2 \leqq \delta$ and thus

$$\operatorname{diam} f_2^{-1}(\mathrm{St}_{K_2}(b)) \leqq \gamma. \tag{1}$$

Consequently,

$$\operatorname{diam} f_1(f_2^{-1}(\mathrm{St}_{K_2}(b))) \leqq \beta, \tag{2}$$

so that the set $f_1(f_2^{-1}(\mathrm{St}_{K_2}(b)))$ is contained in $\mathrm{St}_{K_1}(a)$, for some vertex a of K_1. Assign to every vertex b of K_2 such a vertex a of K_1 and denote it by $\phi(b)$. Thus

$$f_1(f_2^{-1}(\mathrm{St}_{K_2}(b))) \subset \mathrm{St}_{K_1}(\phi(b)), \tag{3}$$

for any vertex b of K_2.

If $b_0, \cdots, b_p$ are vertexes of some simplex of K_2, then their stars in K_2 intersect. We have a fortiori

$$f_1(f_2^{-1}(\mathrm{St}_{K_2}(b_0))) \cap \cdots \cap f_1(f_2^{-1}(\mathrm{St}_{K_2}(b_p))) \neq 0 \tag{4}$$

and, because of (3),

$$\mathrm{St}_{K_1}(\phi(b_0)) \cap \cdots \cap \mathrm{St}_{K_1}(\phi(b_p)) \neq 0. \tag{5}$$

This proves that $\phi(b_0),\cdots,\phi(b_p)$ are vertexes of some simplex of K_1, so that ϕ defines a simplicial mapping $\phi : K_2 \to K_1$. Consequently, $\phi(P_2)$ is the carrier of a subcomplex of K_1 (examples show that $\phi(P_2)$ need not coincide with the carrier $|K_1|$ of K_1 itself).

Given an arbitrary point $x \in X$, the point $f_2(x)$ belongs to some simplex of K_2. If $(b_0,\cdots,b_p)$ is the carrier of $f_2(x)$, then clearly

$$f_2(x) \in \mathrm{St}_{K_2}(b_0) \tag{6}$$

and thus

$$x \in f_2^{-1}(\mathrm{St}_{K_2}(b_0)). \tag{7}$$

Consequently,

$$f_1(x) \in f_1(f_2^{-1}(\mathrm{St}_{K_2}(b_0))) \subset \mathrm{St}_{K_1}(\phi(b_0)). \tag{8}$$

Since $f_1 : X \to P_1$ is a mapping onto P_1, relation (8) proves

$$P_1 = \mathrm{St}_{K_1}(\phi(P_2)). \tag{9}$$

Notice also that (8) implies

$$d_1(f_1(x),\ \phi(b_0)) < \alpha_1. \tag{10}$$

On the other hand, $f_2(x) \in (b_0,\cdots,b_p)$ implies $\phi f_2(x) \in (\phi(b_0),\cdots,\phi(b_p))$, so that

$$d_1(\phi f_2(x),\ \phi(b_0)) \leqq \alpha_1. \tag{11}$$

Finally, (10) and (11) imply

$$d_1(f_1, \phi f_2) \leqq 2\alpha_1. \tag{12}$$

LEMMA 3. *Let K be a finite triangulation and K_0 a subcomplex of K having the property that no vertex of K_0 is a component of K_0. If in addition $\mathrm{St}_K(K_0) = K$, then there exists a continuous map ψ, mapping the carrier $|K_0|$ of K_0 onto the carrier $|K|$ of K, and such that, for any $x \in |K_0|$, x and $\psi(x)$ belong to $\mathrm{Cl}(\mathrm{St}_K(a))$, for some vertex a of K. Consequently, if d denotes the metric on $|K|$ and α the mesh of K, then $d(x, \psi(x)) \leqq 2\alpha$.*

Proof. Let T_i, $i = 1, 2, \cdots, r$, be all the principal(3) simplexes of K, which belong to $K \setminus K_0$. To every T_i assign a simplex $S_{j(i)}$ of K_0 so that T_i and $S_{j(i)}$ have a vertex in common and $S_{j(i)}$ is a principal simplex of K_0. Since $\mathrm{St}_K(K_0) = K$, such a simplex exists and $\dim S_{j(i)} \geqq 1$. Denote by I_j the set of all indices $i \in \{1, \cdots, r\}$ with $j(i) = j$. Given a principal simplex S_j of K_0, choose in its interior as many disjoint closed balls B_i of dimension $\dim S_j$ as there are elements in I_j, $i \in I_j$. Furthermore, choose in the interior U_i of each ball B_i a simple arc L_i.

(3) T is principal provided it is not a proper face of some simplex of K.

Now, define a map $\psi_i : B_i \to S_j \cup T_i$ as follows. On the boundary $B_i \setminus U_i$ of B_i let ψ_i coincide with the identity. On L_i let ψ_i be such that $\psi_i(L_i) = S_j \cup T_i$ (notice that $S_j \cup T_i$ is a Peano continuum). Finally, $T_i \cap S_j$ is a common face of T_i and S_j and it is readily seen that $T_i \cup S_j$ is an absolute retract. Therefore, we can extend the definition of ψ_i from the closed subset $(B_i \setminus U_i) \cup L_i$ of B_i to all of B_i and obtain thus a (continuous) mapping $\psi_i : B_i \to T_i \cup S_j$, which is onto.

Now, define $\psi : |K_0| \to |K|$ as a map coinciding with ψ_i on each ball B_i and being identity elsewhere. Clearly, ψ is continuous and maps $|K_0|$ onto $|K|$, because every principal simplex T_i is contained in $\psi(B_i) = \psi_i(B_i)$. Furthermore, it is clear that, for any $x \in |K_0|$, the points x and $\psi(x)$ belong to $\mathrm{Cl}(\mathrm{St}_K(a))$, for some vertex a of K.

LEMMA 4. *Let X be a continuum, P_1 a polyhedron and $f_1 : X \to P_1$ an ε_1-mapping onto P_1, and let $\delta > 0$ be an arbitrary positive number. Then there is an $\varepsilon_2 > 0$ such that, for any polyhedron P_2 and ε_2-mapping $f_2 : X \to P_2$ onto P_2, there exists a mapping $\pi : P_2 \to P_1$ onto P_1, such that the distance $d_1(f_1, \pi f_2) \leqq \delta$.*

Proof. If P_1 consists of a single point, the assertion is obvious (map P_2 into that point). Therefore, we assume in the following that P_1 is nondegenerate. Choose a triangulation K_1 of P_1 so fine, that its mesh α_1 satisfies

$$\alpha_1 \leqq (1/4)\delta. \tag{13}$$

By Lemma 2 there exists an $\varepsilon_2 > 0$ such that, for any polyhedron P_2 and ε_2-mapping $f_2 : X \to P_2$ onto P_2, there is a mapping $\phi : P_2 \to P_1$ such that $\phi(P_2)$ is the carrier of a subcomplex K_0 of K_1 and that (9) and (12) hold. Clearly, $|K_0| = \phi(P_2)$ cannot consist of a single vertex a_0 of K_1, because (9) would imply $P_1 = \mathrm{St}_{K_1}(a_0)$, and thus a_0 would be the only vertex of K_1. This would mean that P_1 itself consists of the vertex a_0 alone, which contradicts our assumption. On the other hand, X being connected, $|K_0| = \phi(P_2) = \phi f_2(X)$ is connected too. Therefore, a single vertex can never be a component of K_0. This enables us to apply Lemma 3.

We obtain a map $\psi : \phi(P_2) \to P_1$ onto P_1 such that

$$d_1(y, \psi(y)) \leqq 2\alpha_1, \tag{14}$$

for any $y \in \phi(P_2)$.

Now put

$$\pi = \psi\phi. \tag{15}$$

Clearly, π maps P_2 onto P_1 and for any $x \in X$ we have

$$d_1(f_1(x), \pi f_2(x)) \leqq d_1(f_1(x), \phi f_2(x)) + d_1(\phi f_2(x), \psi\phi f_2(x)) \leqq 4\alpha_1 \leqq \delta$$

(by (12), (14) and (13)). Thus

$$d_1(f_1, \pi f_2) \leqq \delta. \tag{16}$$

In order to complete the proof of Theorem 1, we next establish

LEMMA 5. *Let $\{X_i; \pi_{ij}\}$, $i, j = 1, 2, \cdots$, be an inverse sequence of compacta X_i provided with metrics d_i, let X be a compactum with metric d, and let $f_i : X \to X_i$ be ε_i-mappings onto X_i, where $\lim \varepsilon_i = 0$. Let $\delta_i > 0$ be a sequence of real numbers having the following properties*:

(i) *For any set $N_j \subset X_j$ of diameter $\operatorname{diam} N_j \leqq \delta_j$, we have*

$$\operatorname{diam}(\pi_{ij}(N_j)) \leqq \delta_i/2^{j-i},$$

for all $i \leqq j$.

(ii) *$x, x' \in X$ and $d(x, x') \geqq 2\varepsilon_i$ imply $d_i(f_i(x), f_i(x')) > 2\delta_i$.*

(iii) *$d_i(f_i, \pi_{i,i+1} f_{i+1}) \leqq \delta_i/2$, for all i.*

Under these circumstances X is homeomorphic with $X' = \operatorname{Inv} \lim \{X_i; \pi_{ij}\}$.

Proof. First we prove, by induction on $j - i$, that

$$d_i(f_i, \pi_{ij} f_j) \leqq \delta_i, \qquad i \leqq j. \tag{17}$$

For $j - i = 1$ this follows from (iii). Moreover, by the induction hypothesis we have

$$d_{i+1}(f_{i+1}, \pi_{i+1,j} f_j) \leqq \delta_{i+1}, \qquad j - i > 1, \tag{18}$$

which by (i) yields

$$d_i(\pi_{i,i+1} f_{i+1}, \pi_{ij} f_j) \leqq \delta_i/2. \tag{19}$$

This and (iii) give us (17).

Applying (i) to $d_j(f_j, \pi_{jk} f_k) \leqq \delta_j$ we obtain

$$d_i(\pi_{ij} f_j, \pi_{ik} f_k) \leqq \delta_i/2^{j-i}, \qquad i \leqq j \leqq k. \tag{20}$$

Now, we define a map $g_i : X \to X_i$ by setting

$$g_i(x) = \lim_{j \to \infty} \pi_{ij} f_j(x), \qquad x \in X. \tag{21}$$

(20) insures that $\pi_{ij} f_j(x)$, $j = i + 1, i + 2, \cdots$, is a Cauchy sequence, so that g_i is well defined by (21). From (21) and (17) we obtain

$$d_i(f_i(x), g_i(x)) \leqq \delta_i. \tag{22}$$

We can also conclude from (21) that

$$g_i = \pi_{ij} g_j, \qquad i \leqq j, \tag{23}$$

so that $g_i : X \to X_i$ define a map $g : X \to X' = \operatorname{Inv} \lim \{X_i; \pi_{ij}\}$. If $\pi_i : X' \to X_i$ denote as usual natural projections, then, by definition,

$$\pi_i g = g_i. \tag{24}$$

In order to show that $g_i : X \to X$ is continuous, consider an arbitrary number $\mu_i > 0$ and choose j so large that $\delta_i/2^{j-i} \leqq \mu_i/3$, and choose $\nu_i > 0$ in such a manner that $x_1, x_2 \in X$ and $d(x_1, x_2) \leqq \nu_i$ imply $d_j(f_j(x_1), f_j(x_2)) \leqq \delta_j$ (use uniform continuity of f_j). Then, by (i), we obtain

$$d_i(\pi_{ij} f_j(x_1),\ \pi_{ij} f_j(x_2)) \leqq \mu_i 3. \tag{25}$$

On the other hand, by (22), $d_j(f_j(x_k), g_j(x_k)) \leqq \delta_j$, $k = 1,2$, so that (i) and (23) yield

$$d_i(\pi_{ij} f_j(x_k), g_i(x_k)) \leqq \mu_i/3, \qquad k = 1,2. \tag{26}$$

From (25) and (26) we deduce

$$d_i(g_i(x_1),\ g_i(x_2)) \leqq \mu_i, \tag{27}$$

which establishes continuity for g_i. Continuity of g is immediate from the continuity of g_i and (24).

Next, we show that

$$g(X) = X'. \tag{28}$$

X being compact and g continuous, it suffices to prove that $g(X)$ is dense on X'. Let $y \in X'$ and let U be a neighborhood of y in X'. Then there is an index i and some ε-neighborhood U_i about $\pi_i(y)$ such that

$$\pi_i^{-1}(U_i) \subset U. \tag{29}$$

Choose $j > i$ so large that $\delta_i/2^{j-i} < \varepsilon$, and consider the point $\pi_j(y) \in X_j$. Since $f_j : X \to X_j$ is a mapping onto X_j, there is some $x \in X$ such that

$$f_j(x) = \pi_j(y). \tag{30}$$

By (22) we have

$$d_j(\pi_j(y),\ g_j(x)) \leqq \delta_j, \tag{31}$$

and by (i) and (23) it follows

$$d_j(\pi_i(y),\ g_i(x)) \leqq \delta_i/2^{j-i} < \varepsilon. \tag{32}$$

Therefore, $\pi_i g(x) = g_i(x) \in U_i$ and

$$g(x) \in \pi_i^{-1}(U_i) \subset U. \tag{33}$$

This proves that $g(X) \cap U \neq 0$.

Finally, we prove that $g : X \to X'$ is one-to-one. Let $x_1, x_2 \in X, x_1 \neq x_2$, and let i be so large that

$$2\varepsilon_i \leqq d(x_1, x_2) \tag{34}$$

(recall that $\lim \varepsilon_i = 0$). Then $g_i(x_1) = g_i(x_2)$ would imply

$$d_i(f_i(x_1), f_i(x_2)) \leqq 2\delta_i, \tag{35}$$

because of $d_i(f_i(x_k), g_i(x_k)) \leqq \delta_i$, $k = 1, 2$ (see (22)). However, (34) and (35) contradict (ii). Thus $g_i(x_1) \neq g_i(x_2)$ and a fortiori $g(x_1) \neq g(x_2)$. This completes the proof that $g : X \to X'$ is a homeomorphism.

Now we pass to the proof of Theorem 1 itself. On each $P_\alpha \in \Pi$ choose a fixed metric d_α and let d denote a metric on X. We shall define, by induction on i, the following sequences: polyhedra $P_i \in \Pi$, maps $\pi_{i',i} : P_i \to P_{i'}$ onto $P_{i'}$, $i' < i$, real numbers $\varepsilon_i > 0$, such that $\lim \varepsilon_i = 0$, ε_i-mappings $f_i : X \to P_i$ onto P_i and real numbers $\delta_i > 0$; all this in such a manner that (i), (ii) and (iii) of Lemma 5 will be satisfied. Then Lemma 5 will furnish the proof of Theorem 1.

First choose an arbitrary real number $\varepsilon_1 > 0$. By assumption, there is in Π a polyhedron P_1 which admits an ε_1-map $f_1 : X \to P_1$ onto P_1. As δ_1 choose a positive number, such that, $x, x' \in X$ and $d(x, x') \geqq 2\varepsilon_1$ imply $d_1(f_1(x), f_1(x')) > 2\delta_1$. That such a δ_1 exists is readily seen (argument by contradiction).

Now assume that we have already defined P_i, f_i, $\pi_{i',i}$, ε_i and δ_i, for all $i' \leqq i < k$, in accordance with (i), (ii) and (iii), and that $\varepsilon_i < 1/i$. Consider the map $f_{k-1} : X \to P_{k-1}$ and the numbers ε_{k-1} and $\delta_{k-1}/2$. Determine $\varepsilon_k > 0$ by Lemma 4, requiring in addition that $\varepsilon_k < 1/k$. By assumption on X, there is a polyhedron $P_k \in \Pi$ and an ε_k-mapping $f_k : X \to P_k$ onto P_k. Then, by Lemma 4, there exists a mapping $\pi_{k-1,k} : P_k \to P_{k-1}$ onto P_{k-1} such that

$$d_{k-1}(f_{k-1}, \pi_{k-1,k} f_k) \leqq \delta_{k-1}/2. \tag{36}$$

Now consider all the maps $\pi_{ik} : P_k \to P_i$, $i < k$, where $\pi_{ik} = \pi_{i,k-1}\pi_{k-1,k}$. We have a finite collection of uniformly continuous mappings and therefore it is possible to determine a $\delta'_k > 0$ in such a manner that subsets of P_k of diameter not greater than δ'_k map under π_{ik} into subsets of P_i of diameter not greater than $\delta_i/2^{k-i}$. On the other hand, f_k being an ε_k-mapping, there is a number $\delta''_k > 0$, such that $x, x' \in X$ and $d(x, x') \geqq 2\varepsilon_k$ imply $d_k(f_k(x), f_k(x')) > 2\delta'_k$. If we put $\delta_k = \min\{\delta'_k, \delta''_k\}$, we have satisfied all three conditions (i), (ii) and (iii). This concludes the proof.

Remark 3. If we do not require that all π_{ij} in Theorem 1 be onto, then the proof can be simplified and works for disconnected compacta X as well.

Remark 4. A forthcoming paper by one of the authors [15] is devoted to nonmetric Π-like continua X. The main result asserts that X is the inverse limit of some inverse system of metric Π-like continua, but need not be obtainable as the limit of an inverse system of polyhedra from Π.

Question 1. Does Theorem 1 remain valid if one requires that the bonding maps π_{ij} be piecewise linear?

4. **Locally cyclic continua.** In this section we state the definition and certain properties of locally cyclic continua.

A space X is said to be *cyclic* provided no point $x \in X$ is a cut point of X, i.e., $X \setminus \{x\}$ is connected, for each $x \in X$.

A space X is said to be *locally cyclic* provided, for each $x \in X$ and open set U about x, there is an open set V, $x \in V \subset U$, which is a cyclic space.

A point $x \in X$ is said to be a *local cut point* of X provided there exists an open set U, $x \in U$, such that for each open V, $x \in V \subset U$, x is a cut point of V. Notice that cut points of connected spaces X are at the same time local cut points of X. It is readily seen that for locally connected spaces X the property of being locally cyclic is equivalent to the property that no point of X is a local cut point.

EXAMPLE 6. The simple closed curve S^1 is a cyclic continuum, but fails to be locally cyclic.

EXAMPLE 7. Let X be the Sierpiński universal plane curve (obtained by considering a square and by successively deleting first the open middle-ninth of the square, second the open middle-ninths of each of the eight squares remaining, third the open middle-ninths of each of the 64 squares remaining, etc.). Then X is a 1-dimensional locally cyclic continuum which is not an ANR (since it is not LC^1). (See [22].)

LEMMA 6. *If X is a connected and locally cyclic space, then X is also locally connected and cyclic.*

Proof. If X consists of a single point, then the assertion is obviously true. Otherwise, for any neighborhood U about x, the cyclic neighborhood V, $x \in V \subset U$, has the property that $x \in \mathrm{Cl}(V \setminus \{x\})$. Since $V \setminus \{x\}$ is connected, we conclude that so is $V = \{x\} \cup (V \setminus \{x\})$. Hence, X is locally connected.

Now assume that X fails to be cyclic. Then there is a cut point $x_0 \in X$. Therefore, $X \setminus \{x_0\} = U_1 \cup U_2$, where U_1 and U_2 are disjoint nonempty open sets of X. Let V be any open set about x_0, which is cyclic. Clearly, x_0 cannot be a cut point of V. Let $V_i = V \cap U_i$, $i = 1,2$. V_1 and V_2 are open sets of X and $V \setminus \{x_0\} = V_1 \cup V_2$. Therefore, at least one of these sets, say V_1, must be empty. Put $U_1' = U_1$ and $U_2' = U_2 \cup V$. U_1' and U_2' are open sets in X and both are nonempty. Moreover, $X = U_1' \cup U_2'$ and $U_1' \cap U_2' = (U_1 \cap U_2) \cup (U_1 \cap V) = 0$. This gives a contradiction, X being connected.

The following is a notion due to R. L. Wilder [23].

A space X with metric d is said to be 0-*completely avoidable* (0-c.a.) provided, for each $\varepsilon > 0$, there are numbers $\delta > \gamma > 0$, $\varepsilon > \delta$, such that, for any three points $x, y, z \in X$ with $d(x,y) = d(x,z) = \delta$ there exists a connected set $M \subset U(x,\varepsilon) \setminus U(x,\gamma)$ which contains y and z; here $U(x,\varepsilon)$ denotes the ε-neighborhood of x.

LEMMA 7. *Let X be a locally connected* 0-*c.a. continuum. Then for each $\varepsilon > 0$ there exist numbers $\delta_1 > \delta_2 > \gamma > 0$, $\varepsilon > \delta_1$, such that, for any three* *points* x, y, $z \in X$ *with* y, $z \in U(x,\delta_1) \setminus U(x,\delta_2)$, *there exists a connected set* $M \subset U(x,\varepsilon) \setminus U(x,\gamma)$ *which contains* y *and* z.

Proof. X is locally arcwise connected. Therefore, for $\varepsilon > 0$, there is an $\varepsilon_1 > 0$ such that $(3/2)\varepsilon_1 < \varepsilon$, and that any two points x, $y \in X$ with $d(x,y) < (3/2)\varepsilon_1$ can be joined by an arc $L \subset X$ of diameter $\operatorname{diam} L < \varepsilon$. By definition of 0-c.a. spaces (applied to ε_1), it follows that there are numbers $\delta > \gamma > 0$, $\varepsilon_1 > \delta$, such that any two points y', $z' \in X$, with $d(x,y') = d(x,z') = \delta$, can be joined by a connected set $M \subset U(x,\varepsilon_1) \setminus U(x,\gamma) \subset U(x,\varepsilon) \setminus U(x,\gamma)$, because $\varepsilon_1 < \varepsilon$. Notice that $d(x,y) < (3/2)\delta$ implies existence of an arc L, $\operatorname{diam} L < \varepsilon$, joining x and y, for $\delta < \varepsilon_1$.

Now put $\delta_1 = (3/2)\delta$, $\delta_2 = \delta$. Then clearly, $\varepsilon > \delta_1 > \delta_2 > \gamma > 0$. Moreover, if y, z are points of $U(x,\delta_1) \setminus U(x,\delta_2)$, then $d(x,y) < (3/2)\delta$ and, therefore, there is an arc $L \subset U(x,\varepsilon)$ joining x and y. Since $y \in X \setminus U(x,\delta_2)$, and $x \in U(x,\delta_2)$, L must intersect the boundary of $U(x,\delta_2)$. Let y' be the first point on L which lies on this boundary, provided that we consider y as the first end point of L. Let L' be the segment of L between y and y'. Then, clearly, $d(x,y') = \delta_2 = \delta$, L' is a connected set joining y and y' and $L' \subset U(x,\varepsilon) \setminus U(x,\delta) \subset U(x,\varepsilon) \setminus U(x,\gamma)$. In a similar way we define z' on the boundary of $U(x,\delta_2)$ and a connected set $L'' \subset U(x,\varepsilon) \setminus U(x,\gamma)$ joining z and z'. Since $d(x,y') = d(x,z') = \delta$, we have a connected set $M \subset U(x,\varepsilon) \setminus U(x,\gamma)$ joining y' and z'. Consequently, $\tilde{M} = L' \cup M \cup L''$ is a connected set joining y and z, and $\tilde{M} \subset U(x,\varepsilon) \setminus U(x,\gamma)$. This completes the proof of Lemma 7.

The next lemma is important in §5.

LEMMA 8. *Every locally cyclic continuum* X *is* 0-*completely avoidable.*

The proof is based on

LEMMA 9. *Let* X *be a locally connected, locally compact and cyclic space and* F *a compact subset of* X, *and let* $\delta_1 > 0$ *be so small that the closure of the* δ_1-*neighborhood* $U(F,\delta_1)$ *of* F *is compact. Then, for any* $0 < \delta \leqq \delta_1$, *there is a* $\gamma > 0$, $\gamma < \delta$, *having the following property: for any* $x \in F$ *and* y, $z \in X$ *with* $d(x,y) = d(x,z) = \delta$, *there exists a connected set* $M \subset X \setminus U(x,\gamma)$ *joining* y *and* z.

Proof. Assume that the assertion is false. Then we can find a $\delta_0 \leqq \delta_1$ and a sequence $\gamma_n > 0$, $\lim \gamma_n = 0$, and sequences x_n, y_n, z_n of points from X such that $x_n \in F$, $d(x_n, y_n) = d(x_n, z_n) = \delta_0$, but every connected set $M_n \subset X$, joining y_n and z_n, intersects $U(x_n, \gamma_n)$. Since y_n and z_n belong to the compactum $\mathrm{Cl}(U(F,\delta_0))$ (by choice of δ_1), we can assume that $y_n \to y_0 \in X$, $z_n \to z_0 \in X$, and $x_n \to x_0 \in F$. Clearly, $d(x_0, y_0) = d(x_0, z_0) = \delta_0$.

Now we shall prove that whenever a continuum $M \subset X$ contains y_0 and z_0 it also contains x_0.

Indeed, if it were not so, we could find a continuum $M_0 \subset X \setminus \{x_0\}$ containing

y_0 and z_0. Then, clearly, $d(x_0, M_0) > 0$ and thus there is a $\gamma_0 > 0$ with $M_0 \subset X \backslash U(x_0, \gamma_0)$, $\gamma_0 < \delta_0$. Let $U(y_0)$ and $U(z_0)$ be connected neighborhoods of y_0 and z_0 respectively, which do not intersect $U(x_0, \gamma_0)$. For sufficiently large n there are points $y_n \in U(y_0)$, $z_n \in U(z_0)$, $x_n \in U(x_0, \gamma_0/2)$ and we also have $\gamma_n < \gamma_0/2$. Therefore, $U(x_n, \gamma_n) \subset U(x_0, \gamma_0)$ and $M_n = U(y_0) \cup M_0 \cup U(z_0)$ is a connected set joining y_n and z_n and $M_n \cap U(x_n, \gamma_n) \subset M_n \cap U(x_0, \gamma_0) = 0$, which is a contradiction.

Now, $X \setminus \{x_0\}$ is a locally connected, locally compact and connected space, because x_0 cannot be a cut point of X. Hence, by a well-known theorem (see, e.g., [21, (5.2), p. 38]), there is an arc L joining y_0 and z_0 in $X \setminus \{x_0\}$. This, however, contradicts the property that we have just established, according to which each continuum in X through y_0 and z_0 also meets x_0.

Proof of Lemma 8. Given an $\varepsilon > 0$, choose about each $x \in X$ a cyclic open neighborhood $U(x)$ of diameter $\operatorname{diam} U(x) < \varepsilon$. Reduce this open covering to a finite subcovering $\{U_1, \cdots, U_n\}$ and shrink it to another open covering $\{V_1, \cdots, V_n\}$ with $\operatorname{Cl} V_i \subset U_i$ (see, e.g., [14, (33.4), p. 26]). Choose a $\delta > 0$, $\delta < \varepsilon$, so small that $\operatorname{Cl}(U(\operatorname{Cl} V_i, \delta)) \subset U_i$ and that 2δ is smaller than the Lebesgue number of the covering $\{V_1, \cdots, V_n\}$, so that for any subset $A \subset X$ with $\operatorname{diam} A \leqq 2\delta$ there is an index i with $A \subset V_i$. Notice that each U_i is a cyclic, locally connected and locally compact space and that $\operatorname{Cl} V_i \subset U_i$ is compact. Moreover, $\operatorname{Cl}(U(\operatorname{Cl} V_i, \delta)) \subset U_i$ is compact. This enables us to apply Lemma 9 and obtain numbers $\gamma_i > 0$, $\gamma_i < \delta$. Now put $\gamma = \min\{\gamma_1, \cdots, \gamma_n\} > 0$. We shall prove that with this choice of $\delta > \gamma > 0$ we have the conditions required for 0-complete avoidability.

Indeed, let $x \in X$ be any point and let $y, z \in X$ satisfy $d(x,y) = d(x,z) = \delta$. Since $\operatorname{diam} U(x,\delta) \leqq 2\delta$, there is an index i with $U(x,\delta) \subset V_i$. In particular, $x \in \operatorname{Cl} V_i \subset U_i$, $y, z \in \operatorname{Cl} U(x,\delta) \subset \operatorname{Cl} V_i \subset U_i$. By Lemma 9 and the choice of γ, we conclude that there is a connected set $M \subset U_i \setminus U(x, \gamma_i) \subset U_i / U(x, \gamma)$ which contains y and z. Since $\operatorname{diam} U_i < \varepsilon$ and $x \in U_i$, we have also $U_i \subset U(x, \varepsilon)$ and thus $M \subset U(x, \varepsilon) \setminus U(x, \gamma)$.

5. **An imbedding of inverse sequences into products of their terms.** Let $\{X_i; \pi_{ij}\}$ be an inverse sequence of compacta X_i with X as their inverse limit (π_{ij} are onto). If d_i denotes a metric on X_i with $d_i \leqq 1$, then a metric d on $Q = X_1 \times X_2 \times \cdots$, and a fortiori on $X \subset Q$, is given by this relation

$$(1) \qquad d(x, y) = \sum_{i=1}^{\infty} 2^{-i} d_i(\pi_i(x), \pi_i(y)), \qquad x, y \in Q.$$

Following M. K. Fort, Jr. and Jack Segal [9] there is a particular way of imbedding the terms X_j of the inverse sequence into Q. One chooses a fixed point $p_j \in X_j$, for each $j = 1, 2, \cdots$, and defines a mapping $h_j : X_j \to Q$ by the relation

$$h_j(x_j) = (\pi_{1j}(x_j), \pi_{2j}(x_j), \cdots, \pi_{jj}(x_j), p_{j+1}, p_{j+2}, \cdots), \qquad x_j \in X_j, \tag{2}$$

so that one obtains

$$\pi_i h_j(x_j) = \pi_{ij}(x_j), \qquad i \leqq j, \ x_j \in X_j, \tag{3}$$

and

$$\pi_i h_j(x_j) = p_i, \qquad i > j, \ x_j \in X_j. \tag{4}$$

Since $\pi_{jj}(x_j) = x_j$, h_j is a homeomorphism between X_j and $h_j(X_j) = X_j^* \subset Q$.

For $x \in X$, let $x_j = \pi_j(x) \in X_j$ and $x_j^* = h_j(x_j) = h_j\pi_j(x) \in X_j^*$. Then, by (1), (3) and (4) we obtain

$$\begin{aligned} d(x_j^*, x) &= \sum_{i=1}^{j} 2^{-i} d_i(\pi_{ij}(x_j), x_i) + \sum_{i=j+1}^{\infty} 2^{-i} d_i(p_i, x_i) \\ &= \sum_{i=j+1}^{\infty} 2^{-i} d_i(p_i, x_i), \end{aligned} \tag{5}$$

because $\pi_{ij}(x_j) = \pi_{ij}\pi_j(x) = \pi_i(x) = x_i$.

Since $d_i \leqq 1$, we obtain, from (5),

$$d(x_j^*, x) \leqq 2^{-j}, \qquad x \in X, \ x_j^* = h_j\pi_j(x). \tag{6}$$

By a similar computation one can show that

$$d(x_j^*, y_j^*) = \sum_{i=1}^{j} 2^{-i} d_i(x_i, y_i), \qquad x, y \in X, \tag{7}$$

so that

$$|d(x_j^*, y_j^*) - d(x, y)| \leqq 2^{-j}, \quad x, y \in X, \ x_j^* = h_j\pi_j(x), \ y_j^* = h_j\pi_j(y). \tag{8}$$

From $h_j\pi_j(X) = X_j^*$ and (6) it follows that

$$X_j^* \subset U(X, 2^{1-j}) \tag{9}$$

and

$$X \subset U(X_j^*, 2^{1-j}), \tag{10}$$

so that the Hausdorff distance $\rho(X, X_j^*) \leqq 2^{1-j}$ converges to zero. Hence the sets $X_j^* \subset Q$ converge to $X \subset Q$.

It has been proved by M. K. Fort, Jr. and Jack Segal [9] that X_j^* converge 0-regularly(4) towards X if and only if X is locally connected, and it has been proved by Jack Segal [17] that X_j^* converge 0-coregularly(4) towards X if and only if X is semi-locally connected(5) and cyclic.

(4) 0-regular convergence has been introduced by G. T. Whyburn in [20], and 0-coregular convergence by P. A. White in [18].

(5) A continuum X is said to be semi-locally connected provided, for each $x \in X$ and open set $U \subset X$ about x, there is an open set V, $x \in V \subset U$, such that $X \setminus V$ has only a finite number of components. Locally connected continua are always semi-locally connected (see [21, (13.21), p. 20]).

Now, we shall establish a similar result for P. A. White's completely avoidable 0-regular convergence [18].

DEFINITION (P. A. WHITE). Let K be a compactum with metric d and $A_n \subset K$ a sequence of compact subsets of K converging towards a set $A \subset K$. The convergence is said to be completely avoidably 0-regular (0-c.a. regular) provided, for each $\varepsilon > 0$, there are numbers $\delta > \gamma > 0$, $\varepsilon > \delta$, and N such that, for $n \geqq N$ and any $x, y, z \in A_n$ with $d(x,y) = d(x,z) = \delta$, there exists a connected set $M_n \subset A_n \cap [U(x,\varepsilon) \backslash U(x,\gamma)]$ joining y and z.

THEOREM 2. *Let $\{X_i; \pi_{ij}\}$ be an inverse sequence of continua with mappings π_{ij} onto and with the limit $X \subset Q = X_1 \times X_2 \times \cdots$. Let $X_i^* = h_i(X_i) \subset Q$ be the images of X_i under the homeomorphic imbeddings $h_i : X_i \to Q$ given by* (2). *Then, if X is locally cyclic, X_i^* converge* 0-*c.a. regularly to X.*

Proof. Given an $\varepsilon > 0$, put $\varepsilon^* = (2/3)\varepsilon$. By Lemma 8, X is 0-completely avoidable, so that, by Lemma 7, there are numbers $\delta_1^* > \delta_2^* > \gamma^* > 0$, $\varepsilon^* > \delta_1^*$, such that, for any points $u \in X$ and $v, w \in U(u, \delta_1^*) \setminus U(u, \delta_2^*) \subset X$, there exists a connected set

$$M \subset U(u, \varepsilon^*) \setminus U(u, \gamma^*) \tag{11}$$

which contains v and w.

By (8), there exists a number N so large that, for $i \geqq N$, we have

$$|d(h_i\pi_i(u),\ h_i\pi_i(v)) - d(u,v)| < \min\{(1/2)(\delta_1^* - \delta_2^*),\ (1/2)\gamma^*\}, \tag{12}$$

for all $u, v \in X$.

Now put $\delta = (1/2)\ (\delta_1^* + \delta_2^*)$ and $\gamma = (1/2)\gamma^*$. Then clearly,

$$\varepsilon > \varepsilon^* > \delta_1^* > \delta > (1/2)\delta_2^* > (1/2)\gamma^* = \gamma > 0.$$

Now assume that $i \geqq N$ and $x, y, z \in X_i^*$ with $d(x,y) = d(x,z) = \delta$. Since $h_i\pi_i$ is a mapping of X onto X_i^*, there exist points $u, v, w \in X$ such that $h_i\pi_i(u) = x$, $h_i\pi_i(v) = y$, $h_i\pi_i(w) = z$. By (12) we obtain

$$|d(x,y) - d(u,v)| < (1/2)(\delta_1^* - \delta_2^*), \tag{13}$$

and since $d(x,y) = \delta = (1/2)(\delta_1^* + \delta_2^*)$, it follows that $\delta_2^* < d(u,v) < \delta_1^*$. Hence,

$$v \in U(u, \delta_1^*) \setminus U(u, \delta_2^*). \tag{14}$$

The same argument yields

$$w \in U(u, \delta_1^*) \setminus U(u, \delta_2^*). \tag{15}$$

Therefore, v and w can be joined by a connected set M satisfying (11).

Now, consider the connected set $M_i^* = h_i\pi_i(M) \subset X_i^*$. Clearly, y and z belong to M_i^*. Therefore, our proof will be complete if we show that

$$M_i^* \subset U(x,\varepsilon) \setminus U(x,\gamma). \tag{16}$$

Take any point $x' \in M_i^*$ and let $u' \in M$ be such that $h_i\pi_i(u') = x'$. By (11) we have

$$u' \in U(u,\varepsilon^*) \setminus U(u,\gamma^*). \tag{17}$$

This relation and (12) give us

$$\begin{aligned} \gamma &= (1/2)\gamma^* \leqq d(u,u') - (1/2)\gamma^* < d(x,x') \\ &< d(u,u') + (1/2)(\delta_1^* - \delta_2^*) < \varepsilon^* + (1/2)\delta_1^* \\ &< (3/2)\varepsilon^* = \varepsilon, \end{aligned} \tag{18}$$

which proves (16).

Combining Theorem 1 with Theorem 2, or with corresponding results of M. K. Fort, Jr. and Jack Segal ([9] and [17]) we obtain

THEOREM 3. *Let Π be a class of polyhedra and X a Π-like continuum. Then there is a continuum $Q \supset X$ and a sequence of polyhedra $P_n \subset Q$ belonging to the class Π and such that P_n converge towards X. Moreover, if X is*

$$\left\{\begin{array}{l} \textit{locally cyclic} \\ \textit{locally connected} \\ \textit{semi-locally connected and cyclic} \end{array}\right\}$$

then the convergence is

$$\left\{\begin{array}{l} \textit{completely avoidably 0-regular} \\ \textit{0-regular} \\ \textit{0-coregular} \end{array}\right\}$$

respectively.

This theorem relates the property of being Π-like to regular convergence and enables one to exploit known results on regular convergence(6).

6. **Continua which are like closed orientable 2-manifolds.** In this section we combine Theorem 3 with some known results on 0-c.a. regular convergence of closed orientable 2-manifolds, due to P. A. White [18]. First we quote White's Theorem 3.16 of [18].

LEMMA 10 (P. A. WHITE). *If the sequence X_i converges 0-c.a. regularly to*

(6) For a survey of results on regular convergence see [19].

X and each X_i is homeomorphic with the closed orientable 2-manifold M, then X is also a closed orientable 2-manifold and the genus $h(X) \leqq h(M)$(7).

THEOREM 4. *Let M be a closed orientable 2-dimensional manifold and X a locally cyclic continuum which is M-like, i.e., admits ε-maps onto M, for each $\varepsilon > 0$. Then X is homeomorphic to M.*

Proof. By Theorem 3, X is obtainable as the limit of a sequence X_i which converges 0-c.a. regularly to X and each X_i is homeomorphic to M. Then, by Lemma 10, X is a closed orientable 2-manifold of genus $h(X) \leqq h(M)$. To complete the proof, observe that X is an ANR and apply the following result due to T. Ganea [11]:

If X is a compact ANR which is M-like, M being a closed 2-manifold, then X is homeomorphic to M.

However, the same goal can be obtained bypassing Ganea's theorem as follows(8):

X is a closed 2-manifold and a fortiori a compact ANR. Therefore, by a well-known result of S. Eilenberg [6], for a sufficiently small $\varepsilon > 0$, each ε-mapping of X onto M admits a left homotopy inverse, i.e., a mapping $g: M \to X$ such that gf is homotopic to the identity. Furthermore, we know that the homology group $H_2(X, Z) \approx Z \neq 0$. Therefore, it follows, by an argument due to H. Hopf [13], that f induces an isomorphism of the fundamental groups $\pi_1(X) \approx \pi_1(M)$. (See [3, Proposition 2, p. 51].)

However, equality of fundamental groups is a sufficient condition for the homeomorphy of two orientable 2-manifolds. This completes the proof of Theorem 4.

REMARK 5. We believe that the assertion of Theorem 4 is true for closed nonorientable 2-manifolds M as well.

REMARK 6. For 2-manifolds with boundary the assertion of Theorem 4 is false. In fact, let M be a compact 2-manifold with boundary. M can be describe as a 2-sphere from which we have punched out the interiors of $r(M)$ disjoint discs and to which we have attached $h(M)$ handles and $c(M)$ cross-caps (holes closed with Möbius strips). Then we have the following example.

EXAMPLE 8. Let X and M be two compact 2-manifolds and let the boundary of X be nonempty. Then X is M-like provided $h(M) \geqq h(X)$, $c(M) \geqq c(X)$ and $r(M) \geqq r(X) \geqq 1$.

Proof. Let C be one of the $r(X)$ simple closed curves forming the boundary of X. Choose on C two disjoint arcs C_1 and C_2 such that there is an arc on C containing $C_1 \cup C_2$ and having diameter smaller than ε. Orient the two arcs C_1 and C_2 in the opposite way, and identify them, preserving the orientation. We

(7) The genus $h(M)$ is the number of handles of M.

(8) The authors are indebted to Professor I. Berstein for pointing out this alternative.

obtain an ε-mapping onto a new manifold X' whose boundary consists of $r(X') = r(X) + 1$ simple closed curves, one of which has a small counter-image in X. Repeating this process $r(M) - r(X) + 1$ times we obtain a 2-manifold X'', whose boundary consists of $r(X'') = r(M) + 1$ simple closed curves, some of which have very small counter-images in X. Let C be such a curve. Draw about C another simple closed curve C' "concentric" to C and very near C, and denote by A the annulus determined by C and C'. Furthermore, denote by N a closed 2-manifold with $h(N) = h(M) - h(X)$ and $c(N) = c(M) - c(X)$. Clearly, N is obtainable from a disc by suitable identifications on the boundary. Therefore, if N' denotes the manifold obtained from N by punching out the interior of a disc, then N' is obtainable from the annulus A by a mapping $f: A \to N'$, which is one-to-one except on C and is such that it carries C' onto the boundary of N'. This proves that we can remove A from X'' and attach N' on its place, obtaining thus the manifold M and an ε-map of X onto M, for any $\varepsilon > 0$.

REMARK 7. We believe that Example 8 completely describes the class of M-like locally cyclic continua. In other words we have this

CONJECTURE 1. A locally cyclic continuum X, which is like a 2-manifold M with a nonempty boundary, is itself a 2-manifold with a nonempty boundary and is such that $h(X) \leqq h(M)$, $c(X) \leqq c(M)$ and $1 \leqq r(X) \leqq r(M)$.

REMARK 8. It is not possible in Theorem 4 to replace the assumption that X is locally cyclic by the weaker assumption that X be locally connected and cyclic. This follows from the next example.

EXAMPLE 9. The simple closed curve S^1 is like any compact 2-manifold M with the exception of the 2-sphere S^2 and the projective plane P_2.

Proof. Let K be a triangulation of S^1 so fine that every 1-simplex of K is of diameter $< \varepsilon/2$. Let $I_1, \cdots, I_n$ be all the 1-simplexes of K in the cyclic order and let a_i denote the first and b_i the last end point of I_i. Clearly, $a_{i+1} = b_i$, $a_1 = b_n$. Choose a fixed point $p \in S$ and define, for each i, a mapping $f_i : I_i \to I_i \times S^1$, which is onto. One can always obtain in addition that $f_i(a_i) = a_i \times p$ and $f_i(b_i) = b_i \times p$. Now let $f: S^1 \to S^1 \times S^1$ be defined by $f | I_i = f_i$. Because $f_i(b_i) = b_i \times p = a_{i+1} \times p = f_{i+1}(a_{i+1})$, f is well defined and continuous. Clearly, $f(S^1) = S^1 \times S^1$ and for $t \in S^1 \times S^1$ there is an i such that $f^{-1}(t) \subset I_i \cup I_{i+1}$, which proves that f is an ε-mapping of S^1 onto the torus $S^1 \times S^1$.

In order to obtain mappings onto a nonorientable torus (Klein bottle), map $S^1 \setminus \operatorname{Int} I_1 = A$ onto $A \times S^1$ by f, and identify the points $a_1 \times y$ with $b_1 \times \bar{y}$, where $\bar{y} \in S^1$ denotes the antipode of $y \in S^1$. Clearly, we obtain the nonorientable torus as the image of A under a $((3/2)\varepsilon)$-mapping, which is readily extended to all of S^1.

In order to obtain other 2-manifolds (except S^2 and P_2) it suffices to replace f_i, for some $i \neq 1$, by a mapping f_i', which maps I_i onto a suitable 2-manifold with a boundary consisting of at least two simple closed curves; two of the simple closed curves belonging to the boundary are then identified with $a_i \times S^1$ and $b_i \times S^1$ respectively.

In order to show that S^1 is not like the 2-sphere S^2 or the projective plane P_2 it suffices to apply Theorem 1. By this theorem, S^1 would be the inverse limit of 2-spheres or the inverse limit of projective planes. This would imply that $Z = H_1(S^1; Z)$ is the inverse limit of a sequence of trivial groups $0 = H_1(S^2; Z)$ or of groups $Z_2 = H_1(P_2; Z)$, which is impossible (observe that each nonzero element of a limit of groups Z_2 is of order 2).

CONJECTURE 2. Let M be a closed 2-manifold and let X be a 2-dimensional locally connected continuum. If X is M-like, it is homeomorphic with M.

By Theorem 1 and results developed in [9] and [17] we obtain also the following results.

THEOREM 5. *Let X be a locally connected continuum, or a semi-locally connected and cyclic continuum, which is like the simple closed curve S^1. Then X is the simple closed curve.*

THEOREM 6. *Let X be a locally connected continuum which is like the arc I. Then X is the arc I.*

THEOREM 7. *Let X be a locally connected and cyclic continuum which is like the 2-sphere (the 2-cell). Then X is the 2-sphere (the 2-cell).*

THEOREM 8. *Let X be a 2-dimensional locally connected continuum which is like the 2-sphere. Then X is the 2-sphere.*

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